ON THE NUMERICAL SOLUTION OF NON-SMOOTH DIRECT AND INVERSE PROBLEMS

Abstract

The main goal of these lectures is to discuss the numerical solution of direct or inverse problems modelled by partial differential equations or inequalities. The common feature of these problems is that they involve some degree of non-smoothness, via either multivalued operators or non-smooth data or non-existence of solutions. This multiplicity of difficulties suggests that several approaches may be needed to overcome them. Instead of going from the general to the specific, we will take the opposite approach and starting from specific examples derive methods whose applicability goes beyond the examples which generated them. The ***first*** example to be investigated will be the solution of the ***elliptic Monge-Ampère equation*** det $\mathbf{D}^2\psi = f\,(> 0)$, where $\mathbf{D}^2\psi$ is the Hessian matrix of the unknown function ψ. The solution method we will present has the ability to capture classical smooth solutions if they do exist, generalized solutions if classical solutions do not exist and solve the Monge-Ampère equation if function f is replaced by a positive measure (like a positive multiple of a Dirac measure). The ***second*** problem we will investigate originates from image processing: a noisy image being associated with a function f one wants to de-noise it using the so-called ***Elastica model***, that is via the solution of the following non-smooth and non-convex problem from Calculus of Variations

$$u = \arg\min_{v\in H^1(\Omega)}\left[\int_\Omega\left(a+b\left|\nabla\cdot\frac{\nabla v}{|\nabla v|}\right|^2\right)|\nabla v|dx+\frac{1}{2}\int_\Omega|v-f|^2dx\right],$$

with Ω a bounded domain of $\mathbf{R}^2$, a and b two positive coefficients, function f being a representation of the image one wants to denoise. An operator-splitting based solution method will be presented. The ***third*** problem to be discussed will be the following non-smooth nonlinear problem

$$\begin{cases}-\mu\nabla^2 u-\tau_y\nabla\cdot\dfrac{\nabla u}{|\nabla u|}=\lambda e^u \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega,\end{cases}$$

with Ω a bounded domain of $\mathbf{R}^2$, and μ and τ_y two positive constants. If λ is positive, the above problem has the flavor of a (non-smooth nonlinear) eigenvalue problem. Several solution methods will be presented, all based on operator-splitting methods. The graph of a solution is visualized below: the plateau we observe is a consequence of the elliptic multivalued operator $v \rightarrow -\nabla\cdot\dfrac{\nabla v}{|\nabla v|}$.